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A q-Schrödinger algebra, its lowest-weight representations and generalized q-deformed heat/Schrödinger equations

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Abstract. We construct a q-deformation $\hat{\mathcal{S}}_q$ of the centrally-extended Schrödinger algebra $\hat{\mathcal{S}}$ and an algebraic representation theory through lowest-weight representations. We use Verma modules over $\hat{\mathcal{S}}_q$, calculate their singular vectors and factorize the Verma modules by submodules built on the singular vectors. We also give a realization of $\hat{\mathcal{S}}_q$ with q-difference operators and obtain a polynomial realization of the lowest-weight representations and an infinite family of q-difference equations which may be called generalized q-deformed heat/Schrödinger equations. We also apply our methods to the on-shell q-Schrödinger algebra proposed by Floreanini and Vinet.

1. Introduction

Quantum groups attracted much attention about 10 years ago after the seminal papers of Drinfeld [1], Jimbo [2], Faddeev *et al* [3], yet most research is related to the quantum group deformations of simple Lie algebras and groups, whilst there are very few examples of q-deformations of non-semisimple Lie algebras.

We address the latter question in the present paper. We are motivated by the essential role played in physics by non-semisimple Lie algebras; recall, for example, that the quantum mechanics of a free particle in $\mathbb{R}^n$ is governed by the centrally-extended Schrödinger algebra $\hat{\mathcal{S}}(n)$ (for other examples, cf, e.g. [4]). Furthermore, this is interesting because a general deformation theory for non-semisimple Lie algebras is unknown, in general, even in the case when one looks for a q-deformation with q-difference operators for which a Hopf structure may not exist. Usually q-deformations of non-semisimple Lie algebras were obtained by contractions of q-deformations of semisimple Lie algebras (cf the first examples in [5, 6], and, for more references, the recent paper [7]).

In the present paper we give an example of a q-deformation which is not obtained by the standard method of contraction of commutator relations. We give a q-deformation of the centrally-extended Schrödinger algebra in $(1+1)$-dimensional spacetime, and construct and study some of its representations and realizations. (The Schrödinger algebra was introduced for $(3+1)$-dimensional spacetime in [8, 9].) We derive a family of $\hat{\mathcal{S}}_q(1)$ invariant equations, and we call and intrepret its first member as a q-deformed heat/Schrödinger equation. The

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motivation for this is the following. If one performs the same calculations starting with $\hat{\mathcal{S}}(1)$, i.e. construct algebraic representations via Verma modules and factorize by submodules given by singular vectors, one also gets, as explained in [10], a family of equations whose first member is the ordinary heat/Schrödinger equation if one uses the standard vector-field representation of $\hat{\mathcal{S}}(1)$. We note that the somewhat indirect approach in [11] starts with a special q-deformed heat equation and looks for a q-symmetry algebra on its solution variety. The resulting q-deformation of the the Schrödinger algebra in [11], which we shall call *on-shell* deformation in the following, is different from ours and is (expectedly) valid only on the solutions of the q-deformed heat equation under consideration.

The paper is organized as follows. In section 2 we give and explain our q-deformation $\hat{\mathcal{S}}_q$ of the centrally-extended Schrödinger algebra and discuss some of its properties: subalgebras, grading, conjugation. In section 3 we construct the lowest-weight representations of $\hat{\mathcal{S}}_q$. We first construct the Verma modules over $\hat{\mathcal{S}}_q$, find their singular vectors and finally factor the Verma submodules built on the singular vectors. In section 4 we give the vector-field realization of $\hat{\mathcal{S}}_q$ which provides a polynomial realization of the lowest-weight representations constructed in section 3 and an infinite family of q-difference equations which may be called generalized q-deformed heat/Schrödinger equations. In section 5 we apply our methods to the on-shell q-deformation proposed in [11].

2. q-deformed Schrödinger algebra $\hat{\mathcal{S}}_q(1)$

We first recall the classical commutation relations of the centrally-extended Schrödinger algebra $\hat{\mathcal{S}}(1)$ [4]:

$$[P_t, G] = P_x \tag{2.1a}$$
$$[K, P_x] = -G \tag{2.1b}$$
$$[D, G] = G \tag{2.1c}$$
$$[D, P_x] = -P_x \tag{2.1d}$$
$$[D, P_t] = -2P_t \tag{2.1e}$$
$$[D, K] = 2K \tag{2.1f}$$
$$[P_t, K] = D \tag{2.1g}$$
$$[P_x, G] = m. \tag{2.1h}$$

Below in (4.5) we give the standard vector-field realization of $\hat{\mathcal{S}}(1)$.

We use the following q-number notations:

$$[a]_q \doteq \frac{q^a - q^{-a}}{q - q^{-1}} \qquad [a]'_q \doteq [a]_{q^{1/2}} = \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}} = \frac{[a/2]_q}{[1/2]_q} \tag{2.2}$$

and similarly for diagonal operators H instead of a.

Now we construct a q-deformation of the Schrödinger algebra under the following conditions.

(1) A realization of the genarators P_t, P_x, G and K in terms of q-difference operators and multiplication operators should be available.

(2) In the limit $q \to 1$ we should have the classical relations (2.1).

(3) The subalgebra structure should be preserved by the deformation and, in particular, the d-deformed $sl(2, \mathbb{C})$ subalgebra generated by D, K and P_t should coincide with the usual Drinfeld–Jimbo deformation $U_q(sl(2, \mathbb{C}))$.

With these conditions we get for $\hat{\mathcal{S}}_q(1)$ the following non-trivial relations instead of (2.1):

$$P_t G - q G P_t = P_x \tag{2.3a}$$
$$[P_x, K] = G q^{-D} \tag{2.3b}$$
$$[D, G] = G \tag{2.3c}$$
$$[D, P_x] = -P_x \tag{2.3d}$$
$$[D, P_t] = -2P_t \tag{2.3e}$$
$$[D, K] = 2K \tag{2.3f}$$
$$[P_t, K] = [D]_q \tag{2.3g}$$
$$P_x G - q^{-1} G P_x = m \tag{2.3h}$$
$$P_t P_x - q^{-1} P_x P_t = 0. \tag{2.3i}$$

Conditions (2) and (3) can now be checked directly; (2.3*e*)–(2.3*g*) are the standard commutation relations of the Drinfeld–Jimbo deformation $U_q(sl(2,\mathbb{C}))$. Moreover, we obtain a q-deformed centrally-extended Galilei subalgebra generated by P_t, P_x and G. The deformation is a 'mild' one, in the sense that commutators are turned into q-commutators, cf (2.3*a*), (2.3*h*) and (2.3*i*), and it differs from the Galilei algebra q-deformation given in [6], which is not a surprise taking into account that the latter is not a subalgebra of a (q-deformed) Schrödinger algebra. Condition (1) will be discussed in section 4.

The commutation relations (2.3) are graded as the undeformed ones, if we define

$$\deg D = 0 \tag{2.4a}$$
$$\deg G = 1 \tag{2.4b}$$
$$\deg K = 2 \tag{2.4c}$$
$$\deg P_x = -1 \tag{2.4d}$$
$$\deg P_t = -2 \tag{2.4e}$$
$$\deg m = 0. \tag{2.4f}$$

For future reference we also record the following involutive anti-automorphism of the q-Schrödinger algebra valid for *real* q:

$$\begin{aligned} &\omega(P_t) = K \qquad \omega(P_x) = G \qquad \omega(D) = D \\ &\omega(m) = m \qquad \omega(q) = q. \end{aligned} \tag{2.5}$$

3. Lowest-weight modules of $\hat{\mathcal{S}}_q(1)$

Denote as $\mathcal{S}^+ = \mathcal{S}(1)^+$ the subalgebra generated by the positively-graded generators G and K, and as $\mathcal{S}^- = \mathcal{S}(1)^-$ the subalgebra generated by the negatively-graded generators P_x and P_t.

Now we consider lowest-weight modules (LWM) of $\hat{\mathcal{S}}(1)$, in particular, Verma modules, which are standard for semisimple Lie algebras (SSLA) and their q-deformations. A lowest-weight module is characterized by its lowest-weight vector v_0 and its lowest weight. The lowest-weight vector is defined by the property of being annihilated by $\mathcal{S}^-$ and of being an eigenvector of the Cartan generators. The lowest weight is given by the eigenvalues of the Cartan generators on v_0. In our case the Cartan generator is D so that we must have

$$D v_0 = -d v_0 \qquad P_x v_0 = 0 \qquad P_t v_0 = 0 \tag{3.1}$$

where $d \in \mathbb{R}$ will be called the (conformal) weight. (The minus sign is for later convenience.)

We denote by $\mathcal{B}$ the non-positively graded subalgebra generated by D, P_x and P_t. (This is an analogue of a Borel subalgebra.) A Verma module V^d is defined as the LWM with lowest weight $-d$, induced from a one-dimensional representation of $\mathcal{B}$ spanned by v_0, on which the generators of $\mathcal{B}$ act as in (3.1). It is given explicitly by $V^d = U_q(\mathcal{S}^+) \otimes v_0$, where $U_q(\mathcal{S}^+)$ is the q-deformed universal enveloping algebra of $\mathcal{S}^+$. Clearly, $U_q(\mathcal{S}^+)$ has the basis elements $p_{k,\ell} = G^k K^\ell$. The basis vectors of the Verma module are $v_{k,\ell} = p_{k,\ell} \otimes v_0$, (with $v_{0,0} = v_0$). The action of the q-Schrödinger algebraon this basis is derived easily from (2.3):

$$D v_{k,\ell} = (k + 2\ell - d) v_{k,\ell} \tag{3.2a}$$
$$G v_{k,\ell} = v_{k+1,\ell} \tag{3.2b}$$
$$K v_{k,\ell} = v_{k,\ell+1} \tag{3.2c}$$
$$P_x v_{k\ell} = q^{(1-k)/2} m [k]'_q v_{k-1,\ell} + q^{d+1-\ell-k} [\ell]_q v_{k+1,\ell-1} \tag{3.2d}$$
$$P_t v_{k\ell} = [\ell]_q [k + \ell - 1 - d]_q v_{k,\ell-1} + m \frac{[k]'_q [k-1]'_q}{[2]'_q} v_{k-2,\ell}. \tag{3.2e}$$

For the derivation of (3.2) the following relations (which follow from (2.3)) are useful:

$$P_x G^k - q^{-k} G^k P_x = m q^{(1-k)/2} [k]'_q G^{k-1} \tag{3.3a}$$
$$P_x K^\ell - K^\ell P_x = q^{1-\ell} [\ell]_q G K^{\ell-1} q^{-D} \tag{3.3b}$$
$$P_t G^k - q^k G^k P_t = [k]_q G^{k-1} P_x + \frac{[k]'_q [k-1]'_q}{[2]'_q} G^{k-2} \tag{3.3c}$$
$$P_t K^\ell - K^\ell P_t = [\ell]_q K^{\ell-1} [D + \ell - 1]_q. \tag{3.3d}$$

Because of (3.2a) we notice that the Verma module V^d can be decomposed in homogeneous subspaces with respect to grading operator D, (cf (2.4)), as follows:

$$V^d = \oplus_{n=0}^{\infty} V_n^d \tag{3.4a}$$
$$V_n^d = \text{lin.span. } \{v_{k,\ell} | k + 2\ell = n\} \tag{3.4b}$$
$$\dim V_n^d = 1 + \left[\frac{n}{2}\right]_{\text{int}} \tag{3.4c}$$

where $[s]_{\text{int}}$ (not to be confused with $[s]_q$) is the largest integer less than or equal to s.

Next we analyse the reducibility of V^d through analogues of singular vectors. As in the SSLA situation a singular vector v_s in our case is a homogeneous element of V^d, such that $v_s \notin \mathbb{C} v_0$, and

$$P_x v_s = 0 \qquad P_t v_s = 0. \tag{3.5}$$

Now we give the possible singular vectors explicitly. Fix the grade $p > 0$ and denote the singular vector as v_s^p. Consider the case of *even* grade, $p \in 2\mathbb{N}$. Since $v_s^p \in V_p^d$ we have

$$v_s^p = \sum_{\ell=0}^{p/2} a_\ell v_{p-2\ell,\ell} = \mathcal{Q}^p(G, K) \otimes v_0 \qquad p \text{ even.} \tag{3.6}$$

Applying (3.5) we obtain that a singular vector exists only for $d = (p-3)/2$ (as for $q = 1$

[10]) and is given for arbitrary q by the formula

$$\begin{aligned} v_s^p &= a_0 \sum_{\ell=0}^{p/2} (-m[2]'_q)^\ell \binom{p/2}{\ell}_q v_{p-2\ell,\ell} = a_0 (G^2 - m[2]'_q K)_q^{p/2} \otimes v_0 \\ \mathcal{Q}^p(G, K) &= a_0 (G^2 - m[2]'_q K)_q^{p/2} \end{aligned} \tag{3.7}$$

where

$$\binom{p}{s}_q \doteq \frac{[p]_q!}{[s]_q![p-s]_q!} \qquad [n]_q! \doteq [n]_q[n-1]_q \ldots [1]_q. \tag{3.8}$$

For *odd* grade there are no singular vectors as for $q = 1$ [10].

To analyse the consequences of the reducibility of our Verma modules we take the subspace of $V^{(p-3)/2}$:

$$I^{(p-3)/2} = U(\mathcal{S}^+) v_s^p. \tag{3.9}$$

It is invariant under the action of the Schrödinger algebra, and is isomorphic to a Verma module $V^{d'}$ with shifted weight $d' = d - p = -(p+3)/2$. The latter Verma module has no singular vectors, since its weight is restricted from above, $d' \leqslant -\frac{5}{2}$, while it is clear that the necessary weight is greater than or equal to $-\frac{1}{2}$.

Let us denote by $\mathcal{L}^{(p-3)/2}$ the factor module $V^{(p-3)/2}/I^{(p-3)/2}$ and by $|p\rangle$ the lowest-weight vector of $\mathcal{L}^{(p-3)/2}$. As a consequence of (3.5) and (3.7), $|p\rangle$ satisfies

$$P_x|p\rangle = 0 \tag{3.10a}$$

$$P_t|p\rangle = 0 \tag{3.10b}$$

$$\sum_{\ell=0}^{p/2} (-m[2]'_q)^\ell \binom{p/2}{\ell}_q G^{p-2\ell} K^\ell |p\rangle = 0. \tag{3.10c}$$

Now from (3.10*c*) we see that

$$K^{p/2}|p\rangle = -\sum_{\ell=0}^{p/2-1} \frac{1}{(-m[2]'_q)^{p/2-\ell}} \binom{p/2}{\ell}_q G^{p-2\ell} K^\ell |p\rangle. \tag{3.11}$$

By a repeated application of this relation to the basis one can get rid of all powers greater than or equal to $p/2$ of K. Thus the basis of $\mathcal{L}^{(p-3)/2}$ will be a *singleton basis* for $p = 2$, and a *quasi-singleton basis* for $p \geqslant 4$:

$$\dim V_n^{(p-3)/2} = 1 \qquad \text{for } n = 0, 1 \text{ or } n \geqslant p \tag{3.12}$$

and it is given by

$$v_{k\ell}^p \equiv G^k K^\ell |p\rangle \qquad p \in 2\mathbb{N} \qquad k, \ell \in \mathbb{Z}_+ \qquad \ell \leqslant p/2 - 1 \qquad d = (p-3)/2. \tag{3.13}$$

The transformation rules of this basis are (3.2) except (3.2*c*) for $\ell = p/2 - 1$, when we have

$$K v_{k,p/2-1}^p = -\sum_{s=0}^{p/2-1} \frac{1}{(-m[2]'_q)^{p/2-s}} \binom{p/2}{s}_q v_{k+p-2s,s}^p. \tag{3.14c'}$$

From the transformation rules we see that $\mathcal{L}^{(p-3)/2}$ is irreducible. In the simplest case $p = 2$ the irrep $\mathcal{L}^{-1/2}$ is also an irrep of the q-deformed centrally-extended Galilean subalgebra $G_q(1)$ generated by P_x, P_t and G.

Hence, the complete list of the irreducible lowest-weight modules over the q-deformed centrally-extended Schrödinger algebra is given by

- V^d, when $d \neq (p-3)/2,\ p \in 2\mathbb{N}$;
- $\mathcal{L}^{(p-3)/2}$, when $d = (p-3)/2,\ p \in 2\mathbb{N}$.

These irreps are infinite-dimensional.

4. Vector-field realization of $\hat{\mathcal{S}}_q(1)$ and generalized q-deformed heat equations

Let us introduce the 'number' operator N_y for the coordinate $y = x, t$, i.e.

$$N_y y^k = k y^k \tag{4.1}$$

and the q-difference operators $\mathcal{D}_y$ and $\mathcal{D}'_y$, which admit a general definition on a larger domain than polynomials, but on polynomials which are well defined as follows,

$$\mathcal{D}_y \doteq \frac{1}{y}[N_y]_q \tag{4.2a}$$

$$\mathcal{D}'_y \doteq \frac{1}{y[\frac{1}{2}]_q}\left[\frac{N_y}{2}\right]_q = \frac{1}{y}[N_y]'_q \tag{4.2b}$$

so that for any suitable function f we obtain as a consequence of (4.1)

$$\mathcal{D}_y f(y) = \frac{f(qy) - f(q^{-1}y)}{y(q-q^{-1})} \tag{4.3a}$$

$$\mathcal{D}'_y f(y) = \frac{f(q^{\frac{1}{2}}y) - f(q^{-\frac{1}{2}}y)}{y(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}. \tag{4.3b}$$

For $q \to 1$ one has $N_y \to y\partial_y$ and $\mathcal{D}_y, \mathcal{D}'_y \to \partial_y$.

With this notation we find a five-parameter realization of (2.3) via q-difference operators (or vector-field realization for short):

$$P_t = q^{c_1}\mathcal{D}_t q^{(1-c_5)N_t+(1-c_4)N_x} \tag{4.4a}$$

$$P_x = q^{c_2}\mathcal{D}'_x q^{-c_4N_t+(c_3+\frac{1}{2})N_x} \tag{4.4b}$$

$$D = 2N_t + N_x - d \tag{4.4c}$$

$$G = q^{c_2-c_1-c_4+c_5} t\mathcal{D}'_x q^{(c_5-c_4)N_t+(c_3+c_4-\frac{1}{2})N_x} + q^{-c_2-c_3-\frac{1}{2}} mxq^{c_4N_t-(c_3+1)N_x} \tag{4.4d}$$

$$\begin{aligned} K = {} & q^{-c_1+c_5-1+d}t^2\mathcal{D}_t q^{(c_5-1)N_t+c_4N_x} + q^{-c_1+c_5-1+d} tx\mathcal{D}_x q^{(c_5-2)N_t+(c_4-1)N_x} \\ & -q^{-c_1+c_5-1}[d]_q t q^{c_5N_t+c_4N_x} + q^{-2c_2-3c_3-\frac{3}{2}+d}[\tfrac{1}{2}]_q mx^2 q^{2(c_4-1)N_t-2(c_3+1)N_x} \end{aligned} \tag{4.4e}$$

where c_1, c_2, c_3, c_4 and c_5 are arbitrary parameters. (There might be other vector-field realizations that are not equivalent to the one just given.)

For $q = 1$ we recover the standard vector-field realization of $\hat{\mathcal{S}}(1)$, namely,

$$P_t = \partial_t \tag{4.5a}$$

$$P_x = \partial_x \tag{4.5b}$$

$$D = 2t\partial_t + x\partial_x - d \tag{4.5c}$$

$$G = t\partial_x + mx \tag{4.5d}$$

$$K = t^2\partial_t + tx\partial_x - td + (m/2)x^2. \tag{4.5e}$$

Our realization (4.4) may be used to construct a polynomial realization of the irreducible lowest-weight modules considered in section 3. For this case we represent the lowest-weight vector by the function 1. Indeed, the constants in (4.4) are chosen so that (3.1) is satisfied:

$$D1 = -d \qquad P_x 1 = 0 \qquad P_t 1 = 0. \tag{4.6}$$

Applying the basis elements $p_{k,\ell} = G^k K^\ell$ of the universal enveloping algebra $U_q(\mathcal{S}^+)$ to 1 we get polynomials in x, t which will be denoted by $f_{k,\ell} \equiv p_{k,\ell}1$. We have in special cases

$$f_{0,\ell} = q^{\ell^2(c_5-1)/2+\ell(-c_1+(c_5-1)/2)}(-d)^q_\ell t^\ell \times \sum_{s=0}^{\ell} \frac{q^{-s^2(2c_3+2c_4-(c_5/2)+\frac{1}{2})}}{(-d)^q_s} \binom{\ell}{s}_q \left(\frac{q^{\ell(2c_4-c_5-1)+c_1-2c_2-c_3-(c_5/2)+1+d}mx^2}{[2]'_q t}\right)^s \tag{4.7a}$$

$$f_{2k,0} = q^{k^2(c_5-1)/2+k(-c_1+(c_5-1)/2)}(\tfrac{1}{2})^q_k([2]'_q mt)^k \times \sum_{s=0}^{k} \frac{q^{-s^2(2c_3+2c_4-(c_5/2)+\frac{1}{2}}}{(\frac{1}{2})^q_s} \binom{k}{s}_q \left(\frac{q^{k(2c_4-c_5-1)+c_1-2c_2-c_3-(c_5/2)+\frac{1}{2}}mx^2}{[2]'_q t}\right)^s \tag{4.7b}$$

$$f_{2k+1,0} = mxq^{k^2(c_5-1)/2+k(-c_1+c_4+(c_5-3)/2)-c_2-c_3-\frac{1}{2}}(\tfrac{3}{2})^q_k([2]'_q mt)^k \times \sum_{s=0}^{k} \frac{q^{-s^2(2c_3+2c_4-(c_5/2)+\frac{1}{2})}}{(\frac{3}{2})^q_s} \binom{k}{s}_q \left(\frac{q^{k(2c_4-c_5-1)+c_1-2c_2-3c_3-c_4-(c_5/2)-\frac{1}{2}}mx^2}{[2]'_q t}\right)^s \tag{4.7c}$$

where $(a)^q_p$ is the q-Pochhammer symbol

$$(a)^q_p = [a+p-1]_q[a+p-2]_q \dots [a]_q. \tag{4.8}$$

If we choose the constants such that $2c_3 + 2c_4 - c_5/2 = 0$ then the above sums are standard degenerate q-hypergeometric polynomials:

$${}_1F^q_1(-a, b; y) \equiv \sum_{s=0}^{a} \binom{a}{s}_q q^{-(s^2/2)/(b)^q_s}(-y)^s. \tag{4.9}$$

One can show that the basis $f_{k,\ell}$ is a realization of the irreducible lowest-weight representations of $\hat{\mathcal{S}}(1)$ listed at the end of the previous section. Indeed, there is one-to-one correspondence between the states $v_{k,\ell}$ of the Verma modules over $\hat{\mathcal{S}}_q(1)$ and the polynomials $f_{k,\ell}$. The irreducible lowest-weight representations of $\hat{\mathcal{S}}_q(1)$ are factor modules of Verma modules, with factorization over the invariant subspaces generated by singular vectors. This statement is trivial if there is no singular vector. When a singular vector exists, i.e. for the representations $V^{(p-3)/2}$, we obtain a q-difference operator by substituting in $\mathcal{Q}^p(G, K)$ (cf (3.6) and (3.7)) each generator with its vector-field realization. For the irreducibility of $\mathcal{L}^{(p-3)/2}$ it is enough to show that the q-difference operator $\mathcal{Q}^p(G, K)$ vanishes identically when applied to 1. This contains more information as $\mathcal{Q}^p(G, K)$ also gives a q-difference equation invariant under the action of $\hat{\mathcal{S}}_q(1)$. Because of this invariance the solutions of this equation are elements of $\mathcal{L}^{(p-3)/2}$. Thus we have an infinite family of q-difference equations, the family members being labelled by $p \in 2\mathbb{N}$, i.e. we have one equation for each representation space $V^{(p-3)/2}$. These equations may be called generalized q-deformed heat equations (m real) or generalized q-deformed Schrödinger equations (m imaginary). The case $p = 2$ is a q-difference analogue of the ordinary heat/Schrödinger equation.

Before making the last example explicit we make a choice of constants in (4.4) and set for simplicity $c_1 = c_2 = c_3 = c_4 = c_5 = 0$:

$$P_t = \mathcal{D}_t q^{N_t+N_x} \tag{4.10a}$$

$$P_x = \mathcal{D}'_x q^{\frac{1}{2}N_x} \tag{4.10b}$$

$$D = 2N_t + N_x - d \tag{4.10c}$$

$$G = t\mathcal{D}'_x q^{-\frac{1}{2}N_x} + q^{-\frac{1}{2}} mxq^{-N_x} \tag{4.10d}$$

$$K = q^{d-1}t^2\mathcal{D}_t q^{-N_t} + q^{d-1}tx\mathcal{D}_x q^{-2N_t-N_x} - q^{-1}[d]_q t + q^{d-\frac{3}{2}}[\tfrac{1}{2}]_q mx^2 q^{-2N_t-2N_x}. \tag{4.10e}$$

The polynomials from (4.7) simplify to

$$f_{0,\ell} = q^{-\ell(\ell+1)/2}(-d)^q_\ell t^\ell\, {}_1F_1^q\left(-\ell, -d; -\frac{q^{1+d-\ell}mx^2}{[2]'_q t}\right) \tag{4.11a}$$

$$f_{2k,0} = q^{-k(k+1)/2}(\tfrac{1}{2})^q_k ([2]'_q mt)^k\, {}_1F_1^q\left(-k, \tfrac{1}{2}; -\frac{q^{\frac{1}{2}-k}mx^2}{[2]'_q t}\right) \tag{4.11b}$$

$$f_{2k+1,0} = q^{-k(k+3)/2-\frac{1}{2}}(\tfrac{3}{2})^q_k ([2]'_q mt)^k mx\, {}_1F_1^q\left(-k, \tfrac{3}{2}; -\frac{q^{-\frac{1}{2}-k}mx^2}{[2]'_q t}\right). \tag{4.11c}$$

The operator $S_q = \mathcal{Q} = G^2 - [2]'_q mK$ determining the singular vectors reads

$$S_q = t^2 q^{\frac{1}{2}}(\mathcal{D}'^2_x q^{-N_x} - q^{d-\frac{3}{2}}[2]'_q m\mathcal{D}_t q^{-N_t}) + mtx\mathcal{D}'_x([2]'_q - (1+q^{N_x})q^{d-1-2N_t})q^{-\frac{3}{2}N_x}$$
$$+q^{-1}mt(q^{-2N_x} + [2d]'_q) + q^{-2}m^2x^2(1 - q^{d+\frac{1}{2}-2N_t})q^{-2N_x} \tag{4.12}$$

which for $q = 1$ gives

$$S = t^2(\partial_x^2 - 2m\partial_t) + mt(2d+1). \tag{4.13}$$

Hence we interprete for $d = -\frac{1}{2}$ (which corresponds to the lowest singular vector ($p = 2$)) the equation $S_q f = 0$ as a *q-deformed heat/Schrödinger equation* as we motivated in the introduction. The explicit form of this equation is

$$S_q f = 0$$
$$S_q = t^2 q^{\frac{1}{2}}(\mathcal{D}'^2_x q^{-N_x} - q^{-2}[2]'_q m\mathcal{D}_t q^{-N_t}) + mtx\mathcal{D}'_x([2]'_q - (1+q^{N_x})q^{-\frac{3}{2}-2N_t})q^{-\frac{3}{2}N_x}$$
$$-\lambda q^{-1}mtx\mathcal{D}_x q^{-N_x} + \lambda q^{-2}m^2x^2t\mathcal{D}_t q^{-N_t-2N_x} \tag{4.14}$$

where $\lambda \doteq q - q^{-1}$. For $q \mapsto 1$ ($\lambda \mapsto 0$) our equation leads to the ordinary heat/Schrödinger equation.

5. A q-deformed Schrödinger algebra on-shell

We now discuss the q-deformation of the vector-field realization of $\hat{\mathcal{S}}(1)$ given by Floreanini and Vinet [11]. The generators are [11]

$$P_t = \mathcal{D}_t q^{-1-N_t} \tag{5.1a}$$

$$P_x = \mathcal{D}'_x q^{-\frac{1}{2}N_x-\frac{1}{2}} \tag{5.1b}$$

$$D = 2N_t + N_x + \tfrac{1}{2} \tag{5.1c}$$

$$G = t\mathcal{D}'_x q^{-\frac{1}{2}N_x-\frac{3}{2}} + x[\tfrac{1}{2}]_q q^{-N_x-\frac{3}{2}} \tag{5.1d}$$

$$K = t^2\mathcal{D}_t q^{-N_t-2N_x-4} + tx\mathcal{D}'_x q^{-\frac{3}{2}N_x-\frac{5}{2}} + x^2[\tfrac{1}{2}]^2_q q^{-2N_x-4} + t[\tfrac{1}{2}]_q q^{-2N_x-\frac{7}{2}}. \tag{5.1e}$$

Note that the explicit form of the above expressions differs from the one in [11], formulae (9). We change $t \to (1-q^{-2})t$ and $x \to (1-q)x$ (employed in [11] only for the limit $q \to 1$); our definition for the q-difference operators (4.2) is also slightly different; we use N_y instead of $T_y = q^{N_y}$ used in [11]; finally, our generator D is essentially the log of their D.

The advantage of the above form of the q-deformed generators is that the $q \to 1$ limit is more transparent. In that limit we recover (4.5) with $m = \frac{1}{2}$, $d = -\frac{1}{2}$. The value of d

is not accidental since this realization was achieved in [11] as the symmetry algebra of the *solutions* of a q-deformation of the heat equation. Indeed, these generators do not form a closed algebra. We have instead of (2.1)

$$[P_t, G] = P_x q^{-D-\frac{1}{2}} \tag{5.2a}$$
$$q^2 P_x K - K P_x = G \tag{5.2b}$$
$$[D, G] = G \tag{5.2c}$$
$$[D, P_x] = -P_x \tag{5.2d}$$
$$[D, P_t] = -2P_t \tag{5.2e}$$
$$[D, K] = 2K \tag{5.2f}$$
$$[P_t, K] = \frac{1}{[\frac{1}{2}]_q}\left[\frac{D}{2}\right]_q q^{-\frac{3}{2}D-2} - \lambda[\tfrac{1}{4}]_q[\tfrac{3}{4}]_q q^{-2D-2} \tag{5.2g}$$
$$q P_x G - G P_x = [\tfrac{1}{2}]_q q^{-1/2}. \tag{5.2h}$$

However, instead of $[G, K] = 0$ one has

$$[G, K] = L = -\lambda \frac{t^2}{x[\frac{1}{2}]_q}\left[\frac{N_x}{2}\right]_q^2 q^{-2N_x-\frac{7}{2}} - \lambda t x[\tfrac{1}{2}]_q[N_t]_q q^{-N_t-3N_x-\frac{13}{2}} \tag{5.3}$$

where L is a new generator. For our purposes it is enough that the operator L annihilates all functions $f_{k,\ell} = G^k K^\ell 1$. This is the on-shell poperty mentioned in the introduction. In the basis $f_{k,\ell}$ we have

$$f_{0,\ell} = f_{2\ell,0} \tag{5.4a}$$
$$f_{2k,0} = (\tfrac{1}{2})_k^q t^k q_1^{-\frac{1}{2}k^2-3k} F_1^q\left(-k;\, \tfrac{1}{2};\, -\frac{q^{1-k}x^2[\frac{1}{2}]_q^2}{t}\right) \tag{5.4b}$$
$$f_{2k+1,0} = (\tfrac{3}{2})_k^q[\tfrac{1}{2}]_q x t^k q_1^{-\frac{1}{2}(k^2+3)-4k} F_1^q\left(-k;\, \tfrac{3}{2};\, -\frac{q^{-k}x^2[\frac{1}{2}]_q^2}{t}\right) \tag{5.4c}$$

(cf (4.9)). For $q = 1$ these expressions were obtained in [10].

Formula (5.4a) is equivalent to $(G^2 - K)1 = 0$, i.e. we have the q-deformed version of the irrep $\mathcal{L}^{-1/2}$, $(p = 2)$, and the basis consists only of $f_k \equiv f_{k,0} = G^k 1$. The generators act on this basis as follows:

$$D f_k = (k + \tfrac{1}{2}) f_k \tag{5.5a}$$
$$G f_k = f_{k+1} \tag{5.5b}$$
$$K f_k = f_{k+2} \tag{5.5c}$$
$$P_x f_k = k[\tfrac{1}{2}]_q q^{-\frac{3}{2}} f_{k-1} \tag{5.5d}$$
$$P_t f_k = [\tfrac{1}{2}]_q q^{-\frac{5}{2}} b_k f_{k-2} \qquad b_k \doteq \sum_{s=0}^{k-1} s q^{-s} \tag{5.5e}$$

where, by summation convention, $b_0 = b_1 = 0$.

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